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Thermoelastic Analysis of an Elliptical Plate with Heat Source and Radiation type Boundary Conditions

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ABSTRACT: This paper deals with analytical thermoelastic solution in elliptical plate under thermal radiation type boundary conditions on curved surfaces. The Integral transform technique is used to solve the set of equations which are written in dimensionless form. The results are obtained in series form in term of Mathieu functions. Numerical algorithm is developed for numerical computations. Numerical simulated results are depicted graphically for various values of time.

Key words: elliptical plate, Integral transform technique, bending moments, Mathieu functions.

I. INTRODUCTION

McLachlan [11], [12] developed mathematical solution of the heat conduction problem for elliptical cylinder in the form of an infinite Mathieu function series considering special case with neglecting surface resistance. Gupta [5] introduced a finite transform involving Mathieu functions and used for obtaining the solutions of boundary value problem involving elliptic cylinders. Choubey [1] also introduced a finite Mathieu transform whose kernel is given by Mathieu function to solve heat conduction in a hollow elliptic cylinder with radiation.

Kirkpatic *et al*. [8] extended the McLachlan's solution with the involvement of numerical calculation. Erdogdu *et al.* [2], [3] investigated the heat conduction within an elliptical cylinder by means of a finite difference method. Sugano *et al*. [13] dealt with transient thermal stress in a confocal hollow elliptical structures with both face surfaces insulated perfectly and obtained the analytical solution with couple-stresses.

Sato [14] subsequently obtained heat conduction problem of an infinite elliptical cylinder during heating and cooling considering the erect of the surface resistance. Recently El Dhaba [4] used boundary integral method to solve the problem of plane, uncoupled linear thermoelasticity with heat sources for an infinite cylinder with elliptical cross section, subjected to a uniform pressure and to a thermal radiation condition on its boundary. However, there aren't many investigations done or studied to successfully eliminate thermoelastic problems.

In this problem analytical thermoelastic solution in elliptical plate under thermal radiation type boundary conditions on curved surfaces are presented. The Integral transform technique is used to solve the set of equations which are written in dimensionless form. The results presented in series form in term of Mathieu functions.

II. FORMULATION OF THE PROBLEM

In terms of rectangular coordinates x, y the elliptical coordinates ξ, η, are defined by the following transformation $x + i\,y = c(\cosh\xi + i\eta) = c\,coh\;\varsigma$. On equating real and imaginary part using relation between hyperbolic function along with their relation to circular functions resulting rectangular coordinates in terms of elliptical coordinates as

$$x = c\cosh\xi\cos\eta,\; y = c\sinh\xi\sin\eta,\; z = z \;\ldots\; (1)$$

The general heat conduction equation in rectangular coordinate system (x, y, z) is represented as

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{g(x,y,t)}{k}-\frac{1}{\alpha}\frac{\partial\theta(x,y,t)}{\partial t}=0 \qquad \ldots(2)$$

Where $\theta(x, y, t)$ the temperature, *t* is the time and α the thermal diffusivity given by $\alpha = k/\rho C$ with thermal conductivity k, density ρ and specific heat C.

The governing equation of heat conduction with internal heat generation, the initial condition and boundary conditions in elliptical cylindrical coordinates are given, respectively as

$$\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}+\frac{g(\xi,\eta,t)}{k}=\frac{c^2}{2\alpha}(\cosh 2\xi-\cos 2\eta)\frac{\partial\theta}{\partial t} \quad \text{...(3)}$$

$$\theta=f(\xi,\eta,t) \qquad at\ \ t=0,\ \xi_0\le\xi\le\xi_a,\ 0\le\eta\le 2\pi \quad \text{...(4)}$$

$$\theta+k_1\frac{\partial\theta}{\partial\xi}=0 \qquad at\ \ \xi=\xi_0,\ 0\le\eta\le 2\pi,\ t>0 \quad \text{...(5)}$$

$$\theta+k_2\frac{\partial\theta}{\partial\xi}=0 \qquad at\ \ \xi=\xi_a,\ 0\le\eta\le 2\pi,\ t>0 \quad \text{...(6)}$$

$$\theta=0 \qquad at\ \ \eta=0,2\pi\ \ \xi_0\le\xi\le\xi_a\ ,\ t>0 \quad \text{...(7)}$$

where k is thermal conductivity, α is thermal diffusivity of the material of the elliptical cylinder and k_1, k_2 are radiation constants. The equations (2) to (7) constitute the mathematical formulation for temperature change within elliptical cylinder with heat generation under consideration.

Here the cylinder is assumed sufficiently thin and considered free from traction. Since the plate is in a plane stress state without bending. Airy stress function method is applicable to the analytical development of the thermoelastic field. Following Gosh [6] and Jeffry [7], the displacements are given by

$$u=\frac{1}{2\mu}\left(\frac{1}{h}\frac{\partial\gamma}{\partial\eta}-h\frac{\partial\chi}{\partial\xi}\right) \quad \text{...(8)}$$

$$v=\frac{1}{2\mu}\left(\frac{1}{h}\frac{\partial\gamma}{\partial\xi}-h\frac{\partial\chi}{\partial\eta}\right) \quad \text{...(9)}$$

where u,v are displacements components in the directions normal to the curves ξ,η, where $\nabla^2\gamma=0$, and equation (8) and (9) satisfies the equation

$$h^2\nabla^2\chi=-k\theta \quad \text{...(10)}$$

The stress components in terms of χ are given by [14] as

$$\sigma_{\xi\xi}=h^2\frac{\partial^2\chi}{\partial\eta^2}+\frac{c^2h^4}{2}\sinh 2\xi\frac{\partial\chi}{\partial\xi}-\frac{c^2h^4}{2}\sinh 2\eta\frac{\partial\chi}{\partial\eta} \quad \text{...(11)}$$

$$\sigma_{\eta\eta}=h^2\frac{\partial^2\chi}{\partial\xi^2}-\frac{c^2h^4}{2}\sinh 2\xi\frac{\partial\chi}{\partial\xi}+\frac{c^2h^4}{2}\sinh 2\eta\frac{\partial\chi}{\partial\eta} \quad \text{...(12)}$$

$$\sigma_{\xi\eta}=-h^2\frac{\partial^2\chi}{\partial\xi\partial\eta}+\frac{c^2h^4}{2}\sinh 2\eta\frac{\partial\chi}{\partial\xi}+\frac{c^2h^4}{2}\sinh 2\xi\frac{\partial\chi}{\partial\eta} \quad \text{...(13)}$$

where $2h^{-2}=c^2(\cosh 2\xi-\cos 2\eta)$

Equations (10) to (13) constitute the mathematical formulation of displacement and thermal stresses due to temperature changes in elliptical cylinder.

III. SOLUTION FOR THE PROBLEM

To find the temperature function $\theta(\xi,\eta,t)$ we introduce the extended integral transform and its corresponding inversion formula as

$$\bar{\theta}(q_{2m,n},t)=\int_{\xi_0}^{\xi_a}\int_0^{2\pi}(\cosh 2\xi'-\cos 2\eta')\,Ce_{2n}(\xi',q)ce_{2n}(\eta',q)\ \theta(\xi',\eta',t)\,d\xi'\,d\eta' \quad \text{...(14)}$$

$$\theta(\xi,\eta,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\bar{\theta}(q_{2m,n},v_n,t)\,(\cosh 2\xi-\cos 2\eta)\,Ce_{2n}(\xi,q_{2m,n})ce_{2n}(\eta,q_{2m,n}) \quad (15)$$

where ce and Ce represent the Mathieu function and modified Mathieu function respectively.
On applying the integral transform defined in equation (14) to equation (3) to (7) and then using their inversion defined in equation (15), one obtain the expression of temperature at any instant and at all the points of elliptical cylinder as

$$\begin{aligned}\theta(\xi,\eta,t) &= \frac{2}{\xi_0\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-\left(\beta_m{}^2+\nu_n{}^2\right)t}(\cosh 2\xi-\cos 2\eta)\, Ce_{2n}(\xi,q)ce_{2n}(\eta,q)\\ &\times\int_{\xi_0}^{\xi_a}\int_{\eta'=0}^{\eta} f(\xi',\eta')d\xi'd\eta'\\ &+\int_{t'}^{t} e^{-\left(\beta_m{}^2+\nu_n{}^2\right)t}\left(\frac{\alpha}{k}\bar{g}\int_{\xi_0}^{\xi_a}\int_{\eta'=0}^{\eta} d\xi'd\eta' + \alpha\nu_n\int_{\xi_0}^{\xi_a}\phi(\xi',t')d\xi'\right)\end{aligned} \tag{16}$$

where $\quad ce_{2n}(\eta,q)=\sum_{l=0}^{\infty}A_{2l}{}^{(2n)}\cos 2l\eta, \qquad Ce_{2n}(\xi,q)=\sum_{l=0}^{\infty}A_{2l}{}^{(2n)}\cosh 2l\xi$

Assuming Airy's stress function $\chi(\xi,\eta,t)$ which satisfies condition (10) as,

$$\begin{aligned}\chi(\xi,\eta,t) &= \frac{2k\,h^2}{\xi_0\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[e^{-(\beta_m{}^2+\nu_n{}^2)t}/\beta_m{}^2+\nu_n{}^2](\cosh 2\xi-\cos 2\eta)\, Ce_{2n}(\xi,q)ce_{2n}(\eta,q)\\ &\times\int_{\xi_0}^{\xi_a}\int_{\eta'=0}^{\eta} f(\xi',\eta')d\xi'd\eta'\\ &+\int_{t'}^{t} e^{-\left(\beta_m{}^2+\nu_n{}^2\right)t}\left(\frac{\alpha}{k}\bar{g}\int_{\xi_0}^{\xi_a}\int_{\eta'=0}^{\eta} d\xi'd\eta' + \alpha\nu_n\int_{\xi_0}^{\xi_a}\phi(\xi',t')\sin(\beta_m\xi')d\xi'\right)\end{aligned} \tag{17}$$

Now using equations (17) in equations (11) to (13), the stress components are obtained as

$$\begin{aligned}\sigma_{\xi\xi} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\{A_1(t)\,A_2\,[\cosh 2l\xi(\cos 2l\eta((1+l^2)\cos 2\eta - l^2\cosh 2\xi)-2l\sin 2\eta\sin 2l\eta]\\ &+(c^2h^4/2)[-2\cosh 2l\xi\sinh 2l\eta(\cos 2l\eta\sin 2\eta+l(\cos 2\eta-\cosh 2\xi))]\\ &+2\cos 2l\eta\sinh 2\xi[\cosh 2l\xi\sinh 2\xi+l(-\cos 2\eta+\cosh 2\xi)\sinh 2l\xi]\}\end{aligned} \tag{18}$$

$$\begin{aligned}\sigma_{\eta\eta} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\{A_1(t)\,A_2\,(h_2/2)[\cosh 2l\xi\sinh 2\eta\sin 2l\eta 2c^2h^2l(\cos 2\eta-\cosh 2\xi)\\ &+\cos 2l\eta(-8l^2\cos 2\eta+8(1+l^2)\cosh 2\xi+2c^2h^2(\sin 2\eta\sinh 2\eta-\sinh[2\xi]^2))]\\ &+2l\cos 2l\eta\sin 2l\xi\sinh 2l\xi(8+c^2h^2\cos 2\eta-c^2h^2\cosh 2\xi)\}\end{aligned} \tag{19}$$

$$\begin{aligned}\sigma_{\xi\eta} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\{A_1(t)\,A_2\,h^2[\cosh 2l\xi(c^2h^2(\cos 2l\eta\sin 2\eta)+l\sinh 2\eta\sinh 2\eta\\ &\times(\cos 2\eta-\cosh 2\xi))+\sinh 2\xi(4l\sin 2l\eta+c^2h^2\cos 2l\eta\sinh 2\xi)]\\ &+l[-4\cos 2l\eta\sin 2\eta+4l\sin 2l\eta(-\cos 2l\eta+\cosh 2\xi)\\ &+c^2h^2\cos 2l\eta\sinh 2l\xi(-\cos 2l\eta+\cosh 2\xi)]\}\end{aligned} \tag{20}$$

where

$$A_1(t)=\frac{2k\,h^2}{\xi_0\pi}\frac{e^{-\left(\beta_m{}^2+\nu_n{}^2\right)t}}{\beta_m{}^2+\nu_n{}^2},$$

$$\begin{aligned}A_2 &= \int_{\xi_0}^{\xi_a}\int_{\eta'=0}^{\eta} f(\xi',\eta')d\xi'd\eta'\\ &+\int_{t'}^{t} e^{-\left(\beta_m{}^2+\nu_n{}^2\right)t}\left(\frac{\alpha}{k}\bar{g}\int_{\xi_0}^{\xi_a}\int_{\eta'=0}^{\eta} d\xi'd\eta' + \alpha\nu_n\int_{\xi_0}^{\xi_a}\phi(\xi',t')Sin(\beta_m\xi')d\xi'\right).\end{aligned}$$

IV. NUMERICAL RESULTS AND DISCUSSION

Numerical computations have been carried out for Copper metal, with non-dimensional variables as given below.

$$\bar{\theta}=\frac{\theta}{\theta_s}\ ,\quad \bar{\xi}=\frac{\xi}{h}\ ,\quad \bar{\eta}=\frac{\eta}{h},\quad \tau=\frac{\kappa t}{h^2}\ ,\ (\bar{\sigma}_{\xi\xi},\bar{\sigma}_{\eta\eta},\bar{\sigma}_{\xi\eta})=\frac{(\sigma_{\xi\xi},\sigma_{\eta\eta},\sigma_{\xi\eta})}{EG_0\theta_s}$$

with parameters $h=2cm$, $\xi_0=2\ cm$, $\xi_a=4cm$, surrounding temperature θ_s = 32°C, thermal expansion coefficient $\alpha_0=17\times10^{-6}/^oC$, thermal diffusivity $\alpha=1.11\ cm^2/\sec$, shear modulus $G_0=1.866\times10^6\ N/cm^2$, Young's modulus $E=4.963\times10^6\ N/cm^2$.

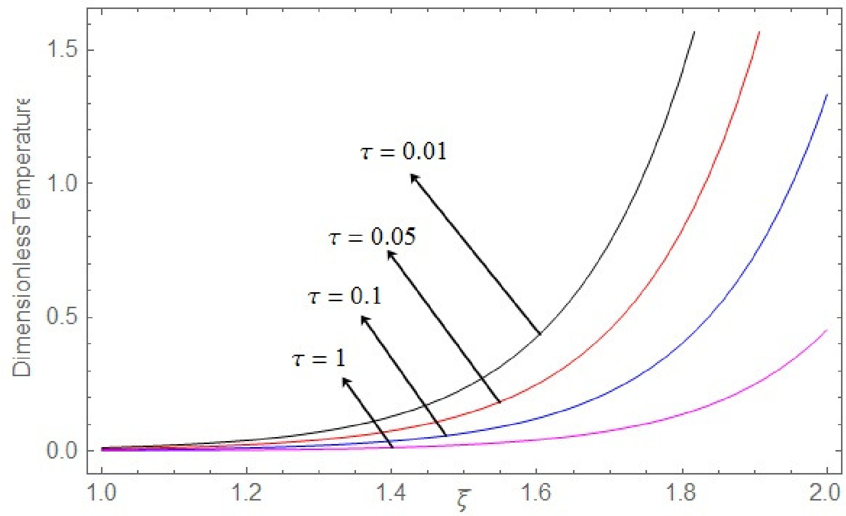

Fig. 1. Variation of dimensionless temperature along ξ direction.

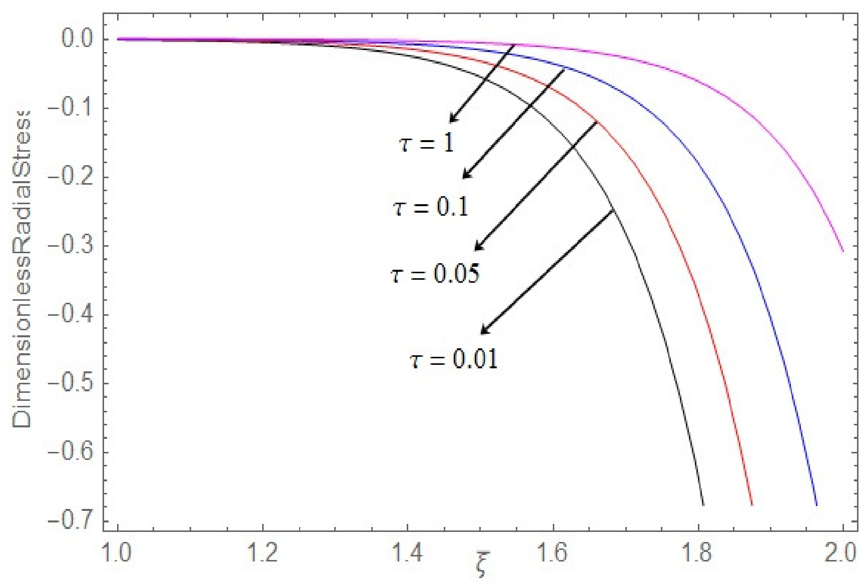

Fig. 2. Variation of dimensionless radial stress along ξ direction.

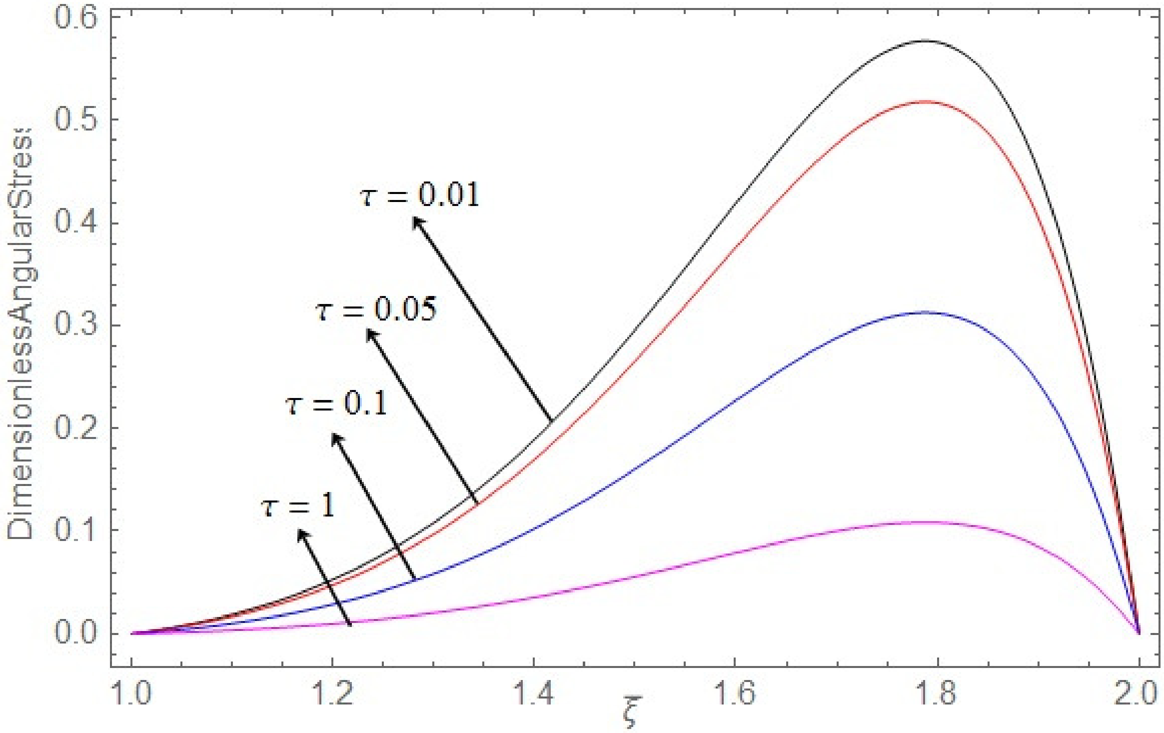

Fig. 3. Variation of dimensionless angular stress along ξ direction.

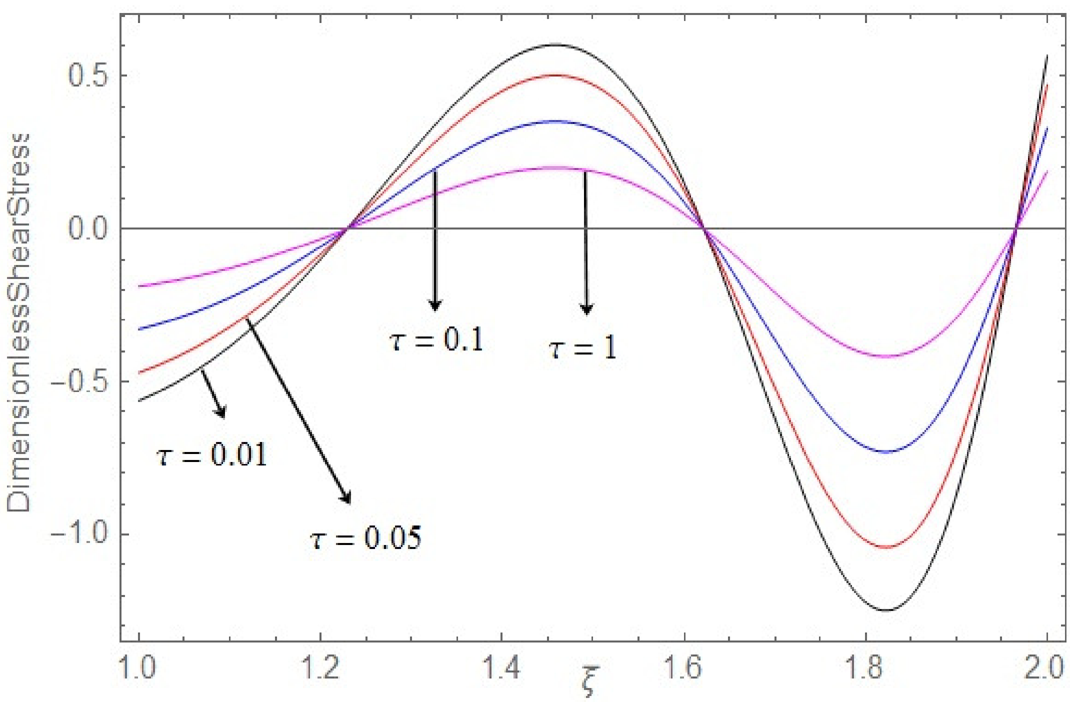

Fig. 4. Variation of dimensionless shear stress along ξ direction.

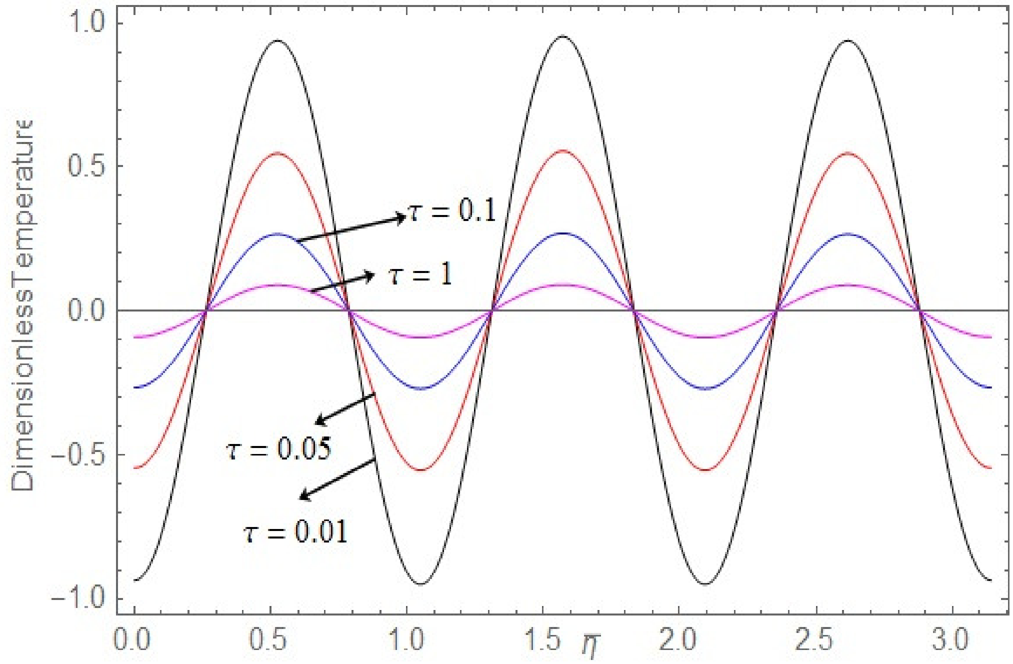

Fig. 5. Variation of dimensionless temperature along η direction.

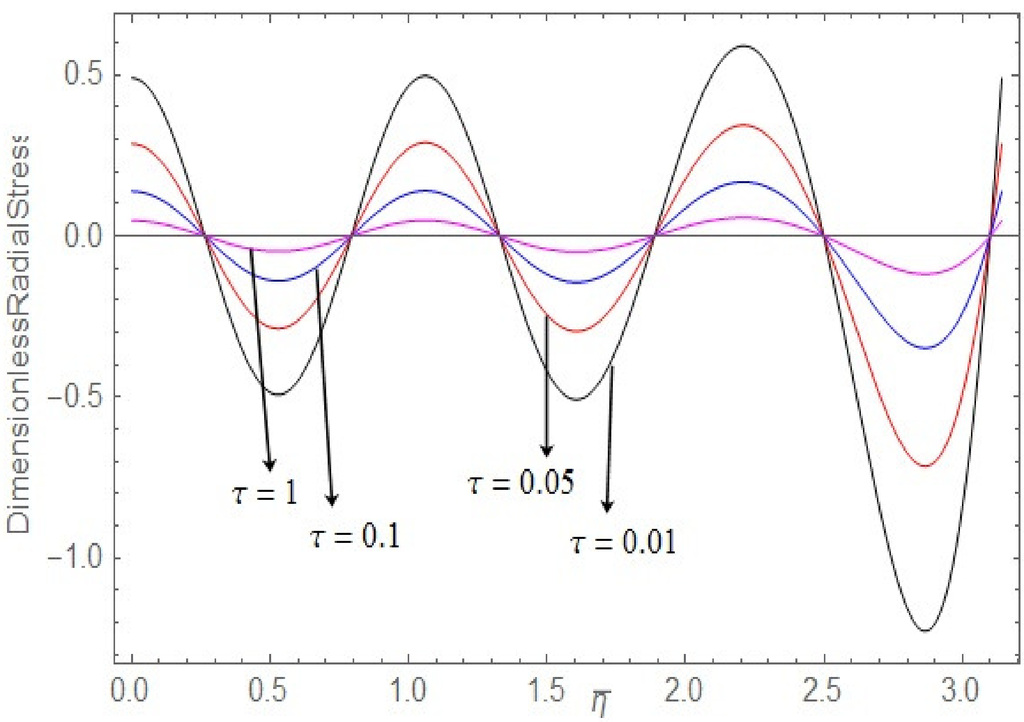

Fig. 6. Variation of dimensionless radial stress along η direction.

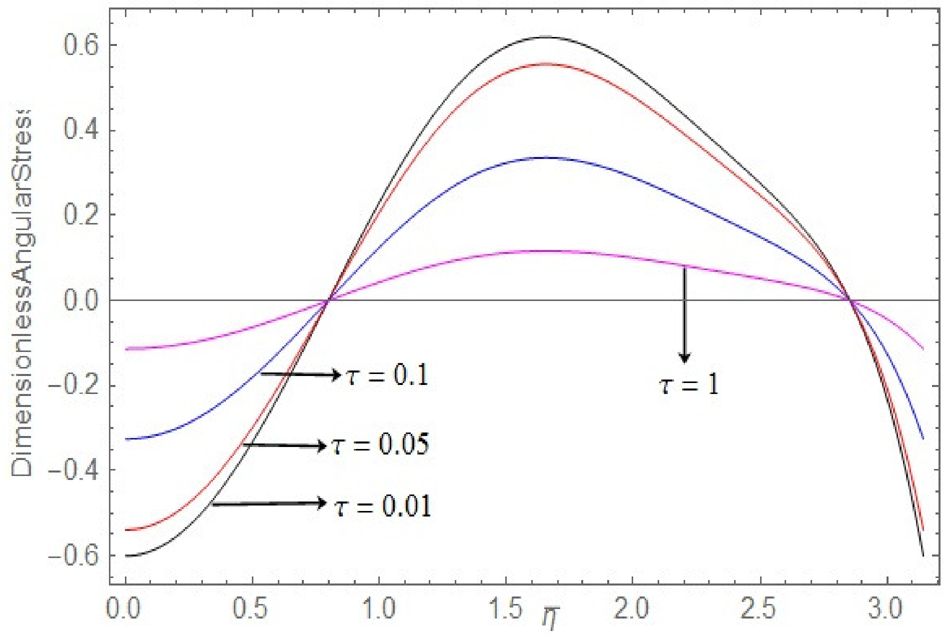

Fig. 7. Variation of dimensionless angular stress along η direction.

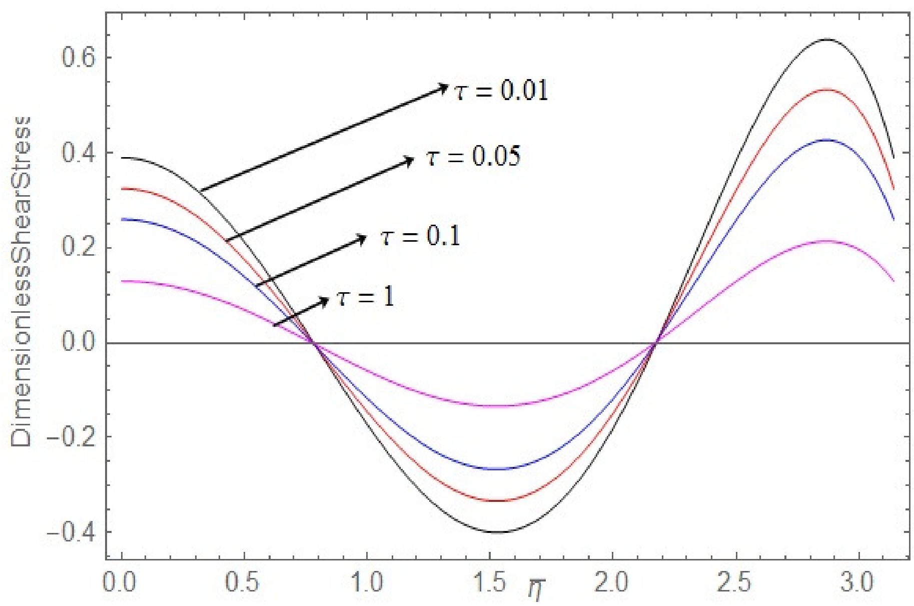

Fig. 8. Variation of dimensionless shear stress along η direction.

V. CONCLUSION

The proposed analytical solution of transient plane thermal stress problem of the confocal elliptical region was handled in elliptical coordinate system. To author's knowledge there have been no reports of solution so far in which sources are generated according to the linear function of the temperature of elliptical plate with boundaries conditions of the radiation type. The analysis of non-stationary two dimensional equation of heat conduction is investigated with the integral transformation method as when there are conditions of radiation type contour acting on the object under consideration. With proposed integral transformation method, it is possible to apply widely to analysis stationary as well as non-stationary temperatures. The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions.

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